

# A GLUING CONSTRUCTION FOR POLYNOMIAL INVARIANTS

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**ABSTRACT.** We give a *polynomial gluing construction* of two groups  $G_X \subseteq GL(\ell, \mathbb{F})$  and  $G_Y \subseteq GL(m, \mathbb{F})$  which results in a group  $G \subseteq GL(\ell + m, \mathbb{F})$  whose ring of invariants is isomorphic to the tensor product of the rings of invariants of  $G_X$  and  $G_Y$ . In particular, this result allows us to obtain many groups with polynomial rings of invariants, including all  $p$ -groups whose rings of invariants are polynomial over  $\mathbb{F}_p$ , and the finite subgroups of  $GL(n, \mathbb{F})$  defined by sparsity patterns, which generalize many known examples.

## 1. INTRODUCTION

The main theme of this paper is the *polynomial gluing construction* described below. Recall that for a finite dimensional vector space  $V$  over an arbitrary field  $\mathbb{F}$ , the symmetric algebra  $S(V)$  is isomorphic to a polynomial algebra, and any subgroup  $G$  of  $GL(V)$  acts on  $S(V)$ .

**Definition 1.1.** Suppose  $V = X \oplus Y$  a direct sum of  $\mathbb{F}$ -subspaces  $X, Y$ ,  $G_X \subseteq GL(X)$ ,  $G_Y \subseteq GL(Y)$  are two groups, and  $\Phi \subseteq \text{Hom}_{\mathbb{F}}(Y, X)$  is a  $(G_X, G_Y)$ -subbimodule. Then one can check

$$(1.1) \quad G = \left\{ g = \begin{bmatrix} g_X & \phi_g \\ 0 & g_Y \end{bmatrix} : \begin{array}{l} g_X \in G_X, \\ g_Y \in G_Y, \\ \phi_g \in \Phi \end{array} \right\}$$

is a subgroup of  $GL(V)$  isomorphic to  $(G_X \times G_Y) \ltimes \Phi$ , where  $g$  acts by

$$g \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g_X(x) + \phi_g(y) \\ g_Y(y) \end{bmatrix}, \text{ for all } \begin{bmatrix} x \\ y \end{bmatrix} \in X \oplus Y.$$

Call  $G$  a *polynomial gluing* of  $G_X$  and  $G_Y$  through  $\Phi$  (or really through  $\pi$ ) if there exists a  $G_X \times G_Y$ -equivariant isomorphism  $\pi : S(V) \rightarrow S(V)^{\Phi}$  of  $\mathbb{F}$ -algebras, where  $\Phi$  acts on  $V$  via the embedding

$$\phi_g \mapsto \begin{bmatrix} 1 & \phi_g \\ 0 & 1 \end{bmatrix}.$$

In particular, this implies  $S(V)^{\Phi}$  is itself a polynomial algebra.

**Proposition 1.2.** *If  $G$  is a polynomial gluing of  $G_X$  and  $G_Y$  through  $\pi$ , then*

$$S(V)^G = \pi \left( S(X)^{G_X} \otimes S(Y)^{G_Y} \right) \quad \text{inside } S(V)^{\Phi}.$$

*In particular, since  $\pi$  is an isomorphism, one has a ring isomorphism*

$$S(V)^G \cong S(X)^{G_X} \otimes S(Y)^{G_Y}$$

*ignoring gradings.*

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*Proof.* Note that  $\Phi$  is a normal subgroup of  $G$  with quotient  $G/\Phi \cong G_X \times G_Y$ . Then

$$\begin{aligned} S(V)^G &= (S(V)^\Phi)^{G_X \times G_Y} = (\pi(S(V)))^{G_X \times G_Y} \\ &= \pi(S(X \oplus Y)^{G_X \times G_Y}) \\ &= \pi(S(X)^{G_X} \otimes S(Y)^{G_Y}) \end{aligned}$$

since  $\pi$  is a  $G_X \times G_Y$ -equivariant isomorphism.  $\square$

**Example 1.3.** Let  $V$  be a vector space over  $\mathbb{F} = \overline{\mathbb{F}}_q$  with basis  $\{x, y\}$ , let  $X = \mathbb{F}x$ ,  $Y = \mathbb{F}y$ , so that  $V = X \oplus Y$ , and define

$$\begin{aligned} G_X &= \mathbb{F}_{q^a}^\times \subseteq GL(X), \\ G_Y &= \mathbb{F}_{q^b}^\times \subseteq GL(Y), \\ \Phi &= \{ \phi : \phi(y) = \gamma x, \gamma \in \mathbb{F}_{q^{ab}} \} \\ &\cong \left\{ \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} : \gamma \in \mathbb{F}_{q^{ab}} \right\} \cong \mathbb{F}_{q^{ab}}, \\ G &= \left\{ \begin{bmatrix} \alpha & \gamma \\ 0 & \beta \end{bmatrix} : \alpha \in \mathbb{F}_{q^a}^\times, \beta \in \mathbb{F}_{q^b}^\times, \gamma \in \mathbb{F}_{q^{ab}} \right\}. \end{aligned}$$

Then one can check that  $G$  is a polynomial gluing of  $G_X$  and  $G_Y$  through  $\Phi$ , with

$$\begin{aligned} \pi : S(V) &\rightarrow S(V)^\Phi, \\ x &\mapsto x, \\ y &\mapsto f, \end{aligned}$$

where  $f = y^{q^{ab}} - x^{q^{ab}-1}y$ , and that

$$\begin{aligned} S(V)^G &= \mathbb{F}[x^{q^a-1}, f^{q^b-1}] \\ &= \pi\left(\mathbb{F}[x^{q^a-1}, y^{q^b-1}]\right) \\ &= \pi(S(X)^{G_X} \otimes S(Y)^{G_Y}). \end{aligned}$$

Proposition 1.2 shows that polynomial gluings are relevant to the *Polynomial Algebra Problem* - determine all finite groups with polynomial invariant rings - for which the non-modular case was answered by the Chevalley–Shephard–Todd theorem and the irreducible case was solved by Kemper and Malle [6]. In the remaining case, that is,  $G$  is reducible and modular, the polynomial gluing construction can provide interesting examples. First note that the construction can be iterated in the following way.

**Definition 1.4.** A group  $G \subseteq GL(V)$  is an *iterated polynomial gluing* of  $G_{X_1}, \dots, G_{X_t}$  if recursively  $G$  is a polynomial gluing of  $G_X$  and  $G_Y$  as above, where  $G_X$  is an iterated polynomial gluing of  $G_{X_1}, \dots, G_{X_s}$  and  $G_Y$  is an iterated polynomial gluing of  $G_{X_{s+1}}, \dots, G_{X_t}$  for some integer  $s$ .

In Section 3 we identify a family of  $p$ -groups given by Nakajima [10] as iterated polynomial gluings. In particular, it implies the following.

**Proposition 1.5.** *A  $p$ -group in  $GL(n, \mathbb{F}_p)$  has a polynomial ring of invariants over  $\mathbb{F}_p$  if and only if it is an iterated polynomial gluing of  $n$  copies of trivial groups  $\{1_{\mathbb{F}_p}\}$ .*

However, there exist groups with polynomial ring of invariants that can not be constructed by polynomial gluing; see Example 3.3.

Another example (which originally motivated the polynomial gluing construction for us) is the following.

**Definition 1.6.** Let  $V$  be the defining representation of  $GL(n, \mathbb{F})$  with basis  $\{x_1, \dots, x_n\}$ . Call a map

$$\sigma : \{1, \dots, n\} \times \{1, \dots, n\} \rightarrow 2^{\mathbb{F}} \setminus \{\emptyset\}$$

a *sparsity pattern*. Given a sparsity pattern  $\sigma$ , define the *sparsity group*  $GL^\sigma(n, \mathbb{F})$  to be the subgroup of  $GL(n, \mathbb{F})$  generated by all transvections

$$\begin{aligned} T_{ij}(a) : x_j &\mapsto x_j + ax_i, \\ x_k &\mapsto x_k, \text{ for } k \neq j \\ \text{with } a &\in \sigma(i, j), \ 1 \leq i \neq j \leq n, \end{aligned}$$

as well as all diagonal matrices

$$\begin{aligned} D_i(a) : x_i &\mapsto ax_i, \\ x_k &\mapsto x_k, \text{ for } k \neq i \\ \text{with } a &\in \sigma(i, i) \setminus \{0\}, \ 1 \leq i \leq n. \end{aligned}$$

For instance, the group discussed in Example 1.3 is  $GL^\sigma(2, \mathbb{F}_{q^{ab}})$  where  $\sigma$  is given by

$$\begin{aligned} \sigma(1, 1) &= \mathbb{F}_{q^a}, \quad \sigma(1, 2) = \mathbb{F}_{q^{ab}}, \\ \sigma(2, 1) &= 0, \quad \sigma(2, 2) = \mathbb{F}_{q^b}. \end{aligned}$$

In Section 4 we investigate the structure of an arbitrary finite sparsity group  $GL^\sigma(n, \mathbb{F})$  and identify it as the following iterated polynomial gluing.

**Theorem 1.7.** *If the sparsity group  $G = GL^\sigma(n, \mathbb{F})$  is finite, then it is an iterated polynomial gluing of subgroups of  $GL(m, \mathbb{F})$  of the following two types:*

- (i) *those with polynomial rings of invariants for  $m = 1, 2$ , and*
- (ii) *those between  $SL(m, \mathbb{F}_q)$  and  $GL(m, \mathbb{F}_q)$  for various integers  $m \in \{3, 4, \dots, n\}$  and finite fields  $\mathbb{F}_q \subseteq \mathbb{F}$ .*

*Consequently the ring of invariants of  $G$  is polynomial.*

This theorem covers many known results asserting that various groups have polynomial invariants, including the following.

- The general linear group  $GL(n, \mathbb{F}_q)$  and the special linear group  $SL(n, \mathbb{F}_q)$  (Dickson [2]),
- The group of all upper triangular matrices with 1's on diagonal (M.-J. Bertin [3]),
- Parabolic subgroups  $G_{n_1, \dots, n_\ell}$  of all block upper triangular matrices with diagonal blocks of size  $n_1, \dots, n_\ell$  (Hewett [4], Mui [8]),
- Seaweed groups  $G_{\alpha, \beta} = G_\alpha \cap (G_\beta)^t$  associated with seaweed Lie algebras [1, 12], where  $\alpha = (\alpha_1, \dots, \alpha_r)$ ,  $\beta = (\beta_1, \dots, \beta_s)$  are integer compositions of  $n$  (Potechin<sup>1</sup>),
- The groups  $G(r)$  of all matrices that agree with  $I_n$  in its first  $r$  rows (Steinberg [14]),
- The groups  $E_{\mathbb{F}_q}(r)$  generated by  $\{T_{in}(\alpha) : \alpha \in \mathbb{F}_q, i = 1, \dots, r\}$  as well as the transpose groups  $E_{\mathbb{F}_q}(r)^t$  for  $r = 1, \dots, n-1$ , (Smith [13, Proposition 8.2.5, 8.2.6]).

*Remark 1.8.* There is another natural way to get sparsity groups in  $GL(n, \mathbb{F})$ . Given a sparsity pattern  $\sigma$ , let

$$GL_\sigma(n, \mathbb{F}) = \{[a_{ij}]_{i,j=1}^n \in GL(n, \mathbb{F}) : a_{ij} \in \sigma(i, j), i, j = 1, \dots, n\}.$$

One can show that  $GL_\sigma(n, \mathbb{F})$  is a group if and only if it equals  $GL^\sigma(n, \mathbb{F})$  and hence it suffices to consider only  $GL^\sigma(n, \mathbb{F})$ ; for this reason we omit the discussion of  $GL_\sigma(n, \mathbb{F})$ .

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## 2. GLUING LEMMA

Using a common technique for showing a ring of invariants to be polynomial we obtain a special kind of polynomial gluing construction. This turns out to be sufficient for our later use in proving Proposition 1.5 and Theorem 1.7.

**Lemma 2.1.** [5, Proposition 16] *Let  $V$  be a vector space over an arbitrary field  $\mathbb{F}$  with basis  $\{x_1, \dots, x_n\}$ , and  $G \subseteq GL(V)$  a finite group. Then  $\mathbb{F}[x_1, \dots, x_n]^G$  is a polynomial algebra if and only if there are algebraically independent homogeneous invariants  $f_1, \dots, f_n \in \mathbb{F}[x_1, \dots, x_n]^G$  such that  $\deg(f_1) \cdots \deg(f_n) = |G|$ .*

**Gluing Lemma.** *Let  $\mathbb{F}_q \subseteq \mathbb{F}$  be a field extension, and consider a finite dimensional  $\mathbb{F}$ -space decomposition  $V = X \oplus Y$  with  $Y$  being  $\mathbb{F}_q$ -rational, i.e.  $Y = Y' \otimes_{\mathbb{F}_q} \mathbb{F}$  for some  $\mathbb{F}_q$ -space  $Y'$ . Let  $X' \subseteq X$  be an  $\mathbb{F}_q$ -subspace stabilized by some subgroup  $G_X \subseteq GL_{\mathbb{F}}(X)$ .*

*Then any subgroup  $G_{Y'} \subseteq GL_{\mathbb{F}_q}(Y')$  gives rise to a polynomial gluing of  $G_X$  and*

$$G_Y = G_{Y'} \otimes 1_{\mathbb{F}}$$

*inside  $GL_{\mathbb{F}}(Y)$  through*

$$\Phi = \text{Hom}_{\mathbb{F}_q}(Y', X') \otimes 1_{\mathbb{F}}$$

*inside  $\text{Hom}_{\mathbb{F}}(Y, X)$ .*

*Proof.* One easily sees  $G_X \circ \Phi \circ G_Y \subseteq \Phi$  from our assumptions and thus (1.1) defines a group  $G$ . Consider the polynomial

$$P(t) = \prod_{x \in X'} (t - x)$$

which is well-known to be an  $\mathbb{F}_q$ -linear function of  $t$  in  $S(X)^{G_X}[t]$  (see, for example, Wilkerson [15]). Let  $\{x_1, \dots, x_\ell\}$  be an  $\mathbb{F}$ -basis for  $X$  and  $\{y_1, \dots, y_m\}$  an  $\mathbb{F}_q$ -basis for  $Y'$  (hence an  $\mathbb{F}$ -basis for  $Y$ ). Define  $f_i = P(y_i)$  in  $S(V)$  for  $i = 1, 2, \dots, m$ . One can check  $x_1, \dots, x_\ell, f_1, \dots, f_m$  are algebraically independent  $\Phi$ -invariants. Also note that

$$\deg(x_1) \cdots \deg(x_\ell) \cdot \deg(f_1) \cdots \deg(f_m) = |X'|^m = |\Phi|.$$

Thus Lemma 2.1 gives

$$\mathbb{F}_q[V]^\Phi = \mathbb{F}_q[x_1, \dots, x_\ell, f_1, \dots, f_m].$$

It follows that the map

$$\begin{aligned} \pi : S(V) &\rightarrow S(V)^\Phi, \\ x_i &\mapsto x_i, & i = 1, \dots, \ell, \\ y_i &\mapsto f_i, & i = 1, \dots, m \end{aligned}$$

is an isomorphism of  $\mathbb{F}_q$ -algebras, and one checks that it is  $G_X \times G_Y$ -equivariant as follows: for every  $(g_X, g_Y) \in G_X \times G_Y$ , one has

$$\pi(g_X x) = g_X x = g_X \pi(x), \text{ for all } x \in X,$$

$$\pi(g_Y y) = P(g_Y y) = g_Y P(y) = g_Y \pi(y), \text{ for all } y \in Y,$$

using the  $\mathbb{F}_q$ -linearity of  $P(t)$ . □

The above lemma is only a special case of the polynomial gluing construction from Definition 1.1, as demonstrated by the following example.

**Example 2.2** (R.E. Stong). Given a prime number  $p$ , let  $\{1, \alpha, \beta\}$  be a basis for  $\mathbb{F} = \mathbb{F}_{p^3}$  over  $\mathbb{F}_p$ . The group  $G$  generated by

$$\begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & \alpha & \beta \\ & 1 & \\ & & 1 \end{bmatrix}$$

naturally acts on  $V = \mathbb{F}^3$  with basis  $\{x_1, x_2, x_3\}$  and has a polynomial ring of invariants [11, §6.3, Example 2]. Let  $X = \mathbb{F}x_1 \oplus \mathbb{F}x_2$  and  $Y = \mathbb{F}x_3$ . It is easy to see that  $G$  is a polynomial gluing of  $G_X = \{1\}$  and  $G_Y = \{I_2\}$ , since the isomorphism  $\pi$  sending  $\{x_1, x_2, x_3\}$  to a set of algebra generators of  $S(V)^G$  is trivially equivariant. But the Gluing Lemma does not apply to it, as one can check  $\Phi \neq \text{Hom}_{\mathbb{F}_q}(Y', X')$  for all possible choices of  $X', Y'$ , and  $\mathbb{F}_q$ .

### 3. THE POLYNOMIAL ALGEBRA PROBLEM

We first consider Nakajima's  $p$ -groups and prove Proposition 1.5.

**Theorem 3.1** (Nakajima [10]). *Let  $\rho : G \hookrightarrow GL(V) = GL(n, \mathbb{F})$  be a representation of a finite  $p$ -group  $G$  over a field  $\mathbb{F}$  of characteristic  $p > 0$ .*

*Suppose that there is a basis  $x_1, \dots, x_n$  for  $V$  such that  $\prod_{i=1}^n |Gx_i| = |G|$  and  $\oplus_{i=1}^j \mathbb{F}x_i$  is an  $\mathbb{F}G$ -submodule of  $V$  for  $j = 1, \dots, n$ . Then  $S(V)^G$  is a polynomial algebra generated by the top Chern classes  $\prod_{x \in Gx_i} x$  of the orbits  $Gx_i$  for  $i = 1, \dots, n$ .*

*Furthermore, the converse holds when  $\mathbb{F} = \mathbb{F}_p$ , that is, if  $G$  is a  $p$ -group inside  $GL(n, \mathbb{F}_p)$  whose ring of invariants is polynomial, then there exists a basis  $x_1, \dots, x_n$  as above.*

**Proposition 3.2.** *Every  $p$ -group  $G$  as in Theorem 3.1 is an iterated polynomial gluing of  $n$  copies of the trivial group  $\{1_{\mathbb{F}}\}$ .*

*Proof.* Since  $\oplus_{i=1}^j \mathbb{F}x_i$  is an  $\mathbb{F}G$ -submodule of  $V$  for  $j = 1, \dots, n$ , all elements in  $G$  are in upper triangular matrix form under  $x_1, \dots, x_n$ . The group  $G_j$  of the  $j$ -th diagonal entries of all elements in  $G$  is canonically embedded in  $G$  and hence has a power of  $p$  as its order. On the other hand, since  $G_j$  is a finite group, the field generated by  $G_j$  must be finite, and so the order of  $G_j$  divides  $p^k - 1$  for some integer  $k$ . This forces  $G_j$  to be trivial. Therefore the diagonal entries of elements in  $G$  are all 1's.

Let

$$H = \{h \in GL(n, \mathbb{F}) : hx_j \in Gx_j, j = 1, \dots, n\}.$$

It is clear that  $G \subseteq H$  and by considering the number of choices of the column vectors of  $h \in H$  we get  $|H| \leq |Gx_1| \cdots |Gx_n| = |G|$ . Therefore  $G = H$  and it follows that the restriction of  $G$  to  $X = \mathbb{F}x_1 \oplus \cdots \oplus \mathbb{F}x_{n-1}$  is

$$G_X = \{g \in GL(n-1, \mathbb{F}) : gx_j \in Gx_j, j = 1, \dots, n-1\}.$$

From this one checks that  $G_X$  satisfies the properties for  $G$  in Theorem 3.1 with  $n$  replaced by  $n-1$ , and thus is an iterated polynomial gluing of  $n-1$  copies of  $\{1_{\mathbb{F}}\}$  by induction.

Let  $Y = \mathbb{F}x_n$  and for each  $g$  in  $G$ , define  $\phi_g \in \text{Hom}_{\mathbb{F}}(Y, X)$  by  $\phi_g(x_n) = gx_n - x_n$ . We wish to show that  $\Phi = \{\phi_g : g \in G\}$  is an  $\mathbb{F}_p$ -subspace of  $\text{Hom}_{\mathbb{F}}(Y, X)$ , i.e.  $\phi_g + \phi_h \in \Phi$  for all  $g, h \in G$ . Since  $G = H$ , there exist  $g', h' \in G$  with  $g'x_j = h'x_j = x_j$  for  $j = 1, \dots, n-1$ , and  $g'x_n = gx_n, h'x_n = hx_n$ . Thus

$$\begin{aligned} g'h'x_n &= g'hx_n \\ &= g'(x_n + \phi_h(x_n)) \\ &= gx_n + g'\phi_h(x_n) \\ &= x_n + \phi_g(x_n) + \phi_h(x_n) \end{aligned}$$

and so  $\phi_g + \phi_h = \phi_{g'h'} \in \Phi$ . Therefore  $\Phi$  is an  $\mathbb{F}_p$ -subspace of  $\text{Hom}_{\mathbb{F}}(Y, X)$  and must be equal to  $\text{Hom}_{\mathbb{F}_p}(Y', X') \otimes_{\mathbb{F}_p} 1_{\mathbb{F}}$ , where  $Y' = \mathbb{F}_p x_n$  and  $X' = \{\phi_g(x_n) : g \in G\}$ , since  $Y'$  is of dimension one. It follows from the Gluing Lemma that  $G$  is a polynomial gluing of  $G_X$  and  $G_Y = \{1_{\mathbb{F}}\}$  and we are done by induction on  $n$ .  $\square$

Combining Theorem 3.1 and Proposition 3.2 one immediately has Proposition 1.5, which gives a simple reason for a  $p$ -group to have a polynomial ring of invariants over  $\mathbb{F}_p$ : the ring of invariants of the trivial group  $\{1_{\mathbb{F}_p}\}$  is polynomial. Of course, it is easy to prove that these  $p$ -groups have polynomial rings of invariants directly by using Lemma 2.1; we do not gain a truly simpler proof by the method of polynomial gluing.

The next example shows that iterating the polynomial gluing construction cannot produce all finite groups with polynomial rings of invariants from the irreducible ones.

**Example 3.3.** The symmetric group  $\mathfrak{S}_n$  acts on  $V$  by permuting the basis  $\{x_1, \dots, x_n\}$ , and the ring of invariants  $S(V)^{\mathfrak{S}_n}$  is always polynomial. One can show that the only nonzero proper subrepresentations of  $V$  are the line  $U$  spanned by  $x_1 + \dots + x_n$  and the hyperplane  $W$  spanned by  $x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n$ . It is clear that  $V = U \oplus W$  if and only if  $\text{char}(\mathbb{F}) = p \nmid n$ . We claim that if  $p \mid n$  then  $\mathfrak{S}_n$  can not be obtained by a polynomial gluing construction unless  $p = n = 2$ .

In fact, when  $p = n = 2$  taking  $X = U = W = \mathbb{F}_2(x_1 + x_2)$  and  $Y = \mathbb{F}_2(x_1 + ax_2)$  for some  $a \neq 1$  one has

$$\mathfrak{S}_2 = \left\{ 1, (12) = \begin{bmatrix} 1 & 1+a \\ 0 & 1 \end{bmatrix} \right\}$$

is a polynomial gluing of two copies of  $\{1\}$ .

Assume for the sake of contradiction  $p \mid n \geq 3$  and  $G = \mathfrak{S}_n$  is a polynomial gluing of  $G_X$  and  $G_Y$ . Then  $X$  is either  $U$  or  $W$ .

If  $X = U$  then  $Y$  is isomorphic to the quotient representation  $V/U$  of  $\mathfrak{S}_n$ , which is easily seen to be faithful. Hence

$$|G_Y| = |G| = |G_X| |G_Y| |\Phi|$$

and thus  $\Phi = \{0\}$ , which implies  $Y$  is a subrepresentation of  $V$ . Then  $Y$  is either  $U$  or  $W$ , which is absurd. However,  $S(Y)^{G_Y}$  is polynomial: one has  $S(V/U)^{\mathfrak{S}_n} = \mathbb{F}[\bar{e}_2, \dots, \bar{e}_n]$  where  $e_i$  is the  $i$ -th elementary symmetric polynomial in  $x_1, \dots, x_n$  and  $\bar{e}_i = e_i + U$ .

If  $X = W$  then the same argument as above leads to a contradiction, since the subrepresentation  $W$  is also faithful. Neither  $S(X)^{G_X} = S(W)^{\mathfrak{S}_n}$  nor  $S(W/U)^{\mathfrak{S}_n}$  is polynomial for  $n \geq 5$  [6, Section 5].

This suggests the following question, for which we currently have no answer.

**Question.** Let  $G \subseteq GL(n, \mathbb{F})$  be a group whose order is divisible by  $\text{char}(\mathbb{F})$  and whose ring of invariants is polynomial. If  $G$  is primitive and has a stable subspace  $X$  of dimension  $m$  in  $V$ , then is there a complement space  $Y$  such that  $V = X \oplus Y$  and  $G$  is a polynomial gluing of  $G_X \subseteq GL(m, \mathbb{F})$  and  $G_Y \subseteq GL(n - m, \mathbb{F})$ ?

#### 4. SPARSITY GROUPS $GL^\sigma(n, \mathbb{F})$

Let  $V$  be the defining representation of  $GL(n, \mathbb{F})$  with basis  $\{x_1, \dots, x_n\}$ . Recall from Definition 1.6 that a sparsity pattern  $\sigma$  gives rise to a group  $GL^\sigma(n, \mathbb{F})$  sitting inside  $GL(n, \mathbb{F})$ . First note that one can order  $x_1, \dots, x_n$  to make all elements in  $GL^\sigma(n, \mathbb{F})$  have a upper triangular block form.

**Definition 4.1.** Define a preorder on  $\{1, \dots, n\}$  by saying  $i \leq_\sigma j$  if there exists a sequence  $i = i_0, i_1, \dots, i_n = j$  such that either  $i_k = i_{k+1}$  or  $\sigma(i_k, i_{k+1}) \neq \{0\}$  for  $k = 0, 1, \dots, n-1$ . This preorder induces an equivalence relation:  $i \sim_\sigma j$  if  $i \leq_\sigma j$  and  $j \leq_\sigma i$ ; denote the equivalence classes by  $J_1, \dots, J_t$ , and their sizes by  $n_1, \dots, n_t$ . By permuting  $x_1, \dots, x_n$  we assume without loss of generality that the preorder “ $\leq_\sigma$ ” is *natural*, i.e.  $i \leq_\sigma j$  only if  $i < j$  or  $i \sim_\sigma j$ . Define  $X_k = \bigoplus_{j \in J_k} \mathbb{F}x_j$ .

**Proposition 4.2.** *Let  $G = GL^\sigma(n, \mathbb{F})$ . Then  $X_1 \oplus \dots \oplus X_k$  is  $G$ -stable for  $k = 1, \dots, t$ ; consequently all elements in  $G$  have the upper triangular block form*

$$(4.1) \quad M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & \cdots & M_{1t} \\ 0 & M_{22} & M_{23} & \cdots & M_{2t} \\ & & \cdots & & \\ & & & \cdots & \\ 0 & 0 & \cdots & 0 & M_{tt} \end{bmatrix}$$

where each  $M_{rs}$  is an  $n_r \times n_s$  matrix. Moreover,  $G$  is irreducible if and only if  $t = 1$ .  $\square$

*Proof.* The assertions are all straightforward except the irreducibility of  $G$  when  $t = 1$ . To see it, assume  $n \geq 2$  and let  $W$  be a nonzero  $\mathbb{F}$ -subspace of  $V$  that is stable under  $G$ . Then  $W$  contains a nonzero element  $v = \sum_{i=1}^n c_i x_i$  with  $c_i \neq 0$  for some  $i$ . Since  $t = 1$  and  $n \geq 2$ , there exists a  $j \neq i$  such that  $\sigma(j, i)$  contains a nonzero element  $a$ . One checks that

$$T_{ji}(a)v - v = (ac_i)x_j \in W$$

which implies  $x_j \in W$  since  $ac_i \in \mathbb{F}^\times$ . Repeating this one has  $x_1, \dots, x_n \in W$ , since  $t = 1$ , and hence  $V = W$ .  $\square$

Now we investigate the structure of a *finite* sparsity group  $G = GL^\sigma(n, \mathbb{F})$  and use it to prove Theorem 1.7. If  $\sigma(i, j) = \{0\}$  whenever  $1 \leq i \neq j \leq n$ , then  $G$  is simply a direct sum of finite subgroups of  $\mathbb{F}^\times$ , and Theorem 1.7 clearly holds in this case.

**Theorem 4.3.** *Assume  $\sigma(i, j) \neq \{0\}$  for some  $(i, j)$  with  $1 \leq i \neq j \leq n$ , and  $G = GL^\sigma(n, \mathbb{F})$  is finite. Then by rescaling basis for  $V$  one can find additive groups  $k_{rs}$  in  $\mathbb{F}$ ,  $1 \leq r, s \leq t$ , such that*

- (a)  $k_{rr}$  is finite field for  $r = 1, \dots, t$ ,
- (b)  $k_{rs} = \{0\}$  if  $r > s$ ,
- (c)  $k_{rs}k_{s\ell} \subseteq k_{r\ell}$  if  $r \leq s \leq \ell$ ,
- (d)  $G = \left\{ [M_{rs}]_{r,s=1}^t : \begin{array}{ll} M_{rr} \in G_{X_r}, & 1 \leq r \leq t, \\ M_{rs} \in M(n_r \times n_s, k_{rs}), & 1 \leq r < s \leq t \end{array} \right\},$

where

- (e)  $SL(n_r, k_{rr}) \leq G_{X_r} \leq GL(n_r, k_{rr})$  unless  $n_r = 2$ ,
- (f)  $M(n_r \times n_s, k_{rs})$  is the space of all  $n_r \times n_s$  matrices with entries in  $k_{rs}$ .

*Proof.* By the assumption  $G$  contains  $T_{ij}(a)$  for some  $a \neq 0$  and thus contains  $T_{ij}(ma) = T_{ij}(a)^m$  for all integers  $m$ . The finiteness of  $G$  forces  $ma = 0$  for some  $m$ , i.e.  $\text{char}(\mathbb{F}) = p > 0$ .

Let

$$S(i, j) = \begin{cases} \{a : T_{ij}(a) \in G\}, & \text{if } 1 \leq i \neq j \leq n, \\ \{a : D_i(a) \in G\}, & \text{if } 1 \leq i = j \leq n. \end{cases}$$

One has  $S(i, j) = 0$  by (4.1) if  $i \in J_r, j \in J_s, r > s$ , and

$$(4.2) \quad S(i, j)S(j, k) \subseteq S(i, k), \quad \text{if } i \neq k$$

by the following equalities:

$$T_{ik}(ab) = \begin{cases} [T_{ij}(a), T_{jk}(b)], & \text{if } i \neq j \neq k, \\ D_i(a)T_{ik}(b)D_i(a^{-1}), & \text{if } i = j \neq k, \\ D_k(b^{-1})T_{ik}(a)D_k(b), & \text{if } i \neq j = k, \end{cases}$$

Define a directed graph on the vertices  $1, 2, \dots, n$  with edges  $i \rightarrow j$  if  $i \neq j$  and  $S(i, j) \neq \{0\}$ . By (4.2), one has the transitivity that  $i \rightarrow j \rightarrow k$  implies  $i \rightarrow k$  if  $i \neq k$ . Hence each  $J_s$  induces a complete subgraph (i.e. a graph with edges in both direction between any pair of distinct vertices); in particular there is a directed cycle passing through all vertices in  $J_s$  (once for each) whenever  $n_s \geq 2$ . Fixing an  $i \in J_r$  with  $r < s$  and letting  $(j, k)$  vary along the edges of this cycle one obtains from (4.2) a corresponding cycle of set-inclusions of the form

$$S(i, j) \cdot a_{jk} \subseteq S(i, k)$$

with nonzero transition scalars  $a_{jk} \in S(j, k)$ . The finiteness of the group  $G$  forces all the above inclusions to be equalities, and one can easily rescale the basis for  $X_s$  to make all but one of the transition scalars  $a_{jk}$  to be 1 so that  $S(i, j) = S(i, k)$  for all  $j, k \in J_s$ . A similar argument clearly works for  $S(i, j)$  with  $i$  varying in  $J_r$  and  $j$  fixed in  $J_s$ ,  $r < s$ . In case  $r = s$  it still works as long as  $i \neq j$  and  $n_r > 2$  so that one is able to choose three distinct indices in (4.2). Combining these together one sees that for any fixed pair  $(r, s)$ ,  $S(i, j)$  takes the same set for all pairs  $(i, j)$  with  $i \in J_r$ ,  $j \in J_s$ ,  $i \neq j$ , unless  $r = s$  and  $n_r = 2$ .

Now we can find the additive groups  $k_{rs}$  and verify the assertions (a)–(f).

It is clear that we should take  $k_{rs} = \{0\}$  if  $r > s$ , so that assertion (b) is true.

If  $r < s$  then let  $k_{rs} = S(i, j)$  for any  $i \in J_r$ ,  $j \in J_s$ , which is clearly closed under addition.

If  $r = s$  then we consider the following cases.

**Case 1.** If  $n_r \geq 3$  then let  $k_{rr} = S(i, j)$  for any  $i, j \in J_r$  with  $i \neq j$ . It is closed under addition since  $T_{ij}(a)T_{ij}(b) = T_{ij}(ab)$ , and closed under multiplication by (4.2) applied to distinct  $i, j, k \in J_r$ . Thus  $k_{rr}$  is a ring (not necessarily unital), and the finiteness of  $G$  forces  $k_{rr}$  to be a finite field. Hence (a) holds for this case.

Then one has  $1 \in S(i, j)$  and  $S(j, j) \subseteq S(i, j) = k_{rr}$  by (4.2). It follows from Lemma 4.4 below that  $G_{X_r}$  consists of all  $n_r \times n_r$  matrices with determinant in the multiplicative group generated by  $\cup_{j \in J_r} \sigma(j, j) \setminus \{0\}$ . Hence (e) holds for this case.

**Case 2.** If  $n_r = 2$ , i.e.  $J_r = \{i, j\}$ , then let  $k_{rr}$  be the field generated by  $S(i, j)$  and  $S(j, i)$ . To show it is finite, it suffices to show any nonzero element  $a \in S(i, j) \cup S(j, i)$  satisfies a polynomial equation. By the rescaling of basis for  $X_r$  that has been used before, one may assume  $1 \in S(i, j)$  and  $0 \neq a \in S(j, i)$ , without loss of generality. By induction one sees that the  $(j, i)$ -entry of  $(T_{ij}(1)T_{ji}(a))^d$  is a monic polynomial  $f_d(a)$  of degree  $d$ , and the finiteness of  $G$  forces  $f_d(a) = a$  for a sufficiently large  $d$ . Therefore (a) is true for this case.

**Case 3.** If  $J_r = \{i\}$  then let  $k_{rr}$  be the field generated by  $\sigma(i, i)$ , which is finite since  $\text{char}(\mathbb{F}) > 0$  and  $G$  is finite. It is obvious that the assertions (a) and (e) are true in this case.

It follows easily from (4.2) and the above definition of  $k_{rr}$  that (c) holds:  $k_{rs}k_{s\ell} \subseteq k_{r\ell}$ .

It remains to show (d) and (f). The inclusion “ $\subseteq$ ” in (d) is clear (consider the generators of  $G$ ). Conversely, note that the right hand side of (d) is generated by all  $D_r(M_{rr})$  and  $T_{rs}(M_{rs})$  with  $M_{rr} \in G_{X_r}$ ,  $M_{rs} \in M(n_r \times n_s, k_{rs})$ ,  $1 \leq r < s \leq t$ , where  $D_r(M_{rr})$  and  $T_{rs}(M_{rs})$  both have the block form (4.1) such that the  $r$ -th diagonal block of  $D_r(M_{rr})$  is  $M_{rr}$ , the  $(r, s)$ -block of  $T_{rs}(M_{rs})$  is  $M_{rs}$ , and all other blocks are either zero (if off diagonal) or identity (if on diagonal). It suffices to show all these  $D_r(M_{rr})$  and  $T_{rs}(M_{rs})$  belong to  $G$ .

If  $M_{rr} \in G_{X_r}$  then  $G$  contains an element  $M$  whose  $r$ -th diagonal block is  $M_{rr}$ . Writing  $M$  as a product of the generators for  $G$ , one sees that the only way to alter the  $r$ -th diagonal



block is to multiply by  $T_{ij}(a)$  or  $D_i(a)$  with  $i, j$  both in  $J_r$ . Hence  $D_r(M_{rr})$  is a product of these generators and must belong to  $G$ .

If  $M_{rs} \in M(n_r \times n_s, k_{rs})$  then  $T_{rs}(M_{rs}) = [a_{ij}]_{i,j=1}^n$  is a product of  $T_{ij}(a_{ij})$  as  $i, j$  run through  $J_r, J_s$ , respectively. The definition of  $k_{rs} = S(i, j)$  implies that all  $T_{ij}(a_{ij})$  belong to  $G$  and so does the product  $T_{rs}(M_{rs})$ .

We have verified all the assertions and the proof of the theorem is now complete.  $\square$

**Lemma 4.4.** *If  $K$  is a subgroup of  $\mathbb{F}^\times$  generated by  $S$ , then the group  $G$  of all  $n \times n$  matrices with entries in  $\mathbb{F}$  and determinant in  $K$  is generated by all transvections  $T_{ij}(a)$  with  $a \in \mathbb{F}$  and  $1 \leq i \neq j \leq n$  as well as all diagonal matrices  $D_i(a)$  with  $a \in S$  and  $1 \leq i \leq n$ .*

*Proof.* It is well known that  $SL(n, \mathbb{F})$  is generated by the transvections  $T_{ij}(a)$  with  $a \in \mathbb{F}$  and  $1 \leq i \neq j \leq n$  (see, for example, [7]), and hence contained in the group  $H$  generated by these transvections together with the diagonal matrices  $D_i(a)$  with  $a \in S$  and  $1 \leq i \leq n$ . One also has  $D_1(b) \in H$  for any  $b \in K$ . Therefore  $H = G$ .  $\square$

*Proof of Theorem 1.7.* It is clear from Theorem 4.3 that the Gluing Lemma in Section 2 applies to  $G$  with  $\mathbb{F}_q = k_{tt}$  and

$$\begin{aligned} X &= X_1 \oplus \cdots \oplus X_{t-1}, \\ Y &= X_t, \\ Y' &= \bigoplus_{j \in J_t} k_{tt} x_j, \\ X' &= \bigoplus_{r=1}^{t-1} \bigoplus_{i \in J_r} k_{rt} x_i. \end{aligned}$$

By induction on  $t$  one shows that  $G$  is an iterated polynomial gluing of  $G_{X_1}, \dots, G_{X_t}$ . Then Theorem 1.7 follows from Proposition 1.2, Theorem 4.3 (e), and the following result of Nakajima (see also Kemper and Malle [6, Proposition 7.1]).  $\square$

**Theorem 4.5.** [9, Theorem 5.1] *If  $G \subseteq GL(2, \mathbb{F})$  is a finite group generated by pseudoreflections then  $S(V)^G$  is polynomial.*

*Remark 4.6.* In general assertion (e) in Theorem 4.3 does not hold for the case  $n_r = 2$ . Let  $\mathbb{F} = \mathbb{F}_4$  and  $a$  an element in  $\mathbb{F}_4 \setminus \mathbb{F}_2$ . Define  $\sigma$  by

$$\begin{aligned} \sigma(1, 1) &= \mathbb{F}_2, & \sigma(1, 2) &= \{a\}, \\ \sigma(2, 1) &= \{1\}, & \sigma(2, 2) &= \mathbb{F}_2. \end{aligned}$$

Thus  $t = 1$  here. It is easy to calculate  $|GL^\sigma(2, \mathbb{F}_4)| = 10$ . This rules out  $k_{11} = \mathbb{F}_2$  in assertion (e) since  $|GL(2, \mathbb{F}_2)| = 6$ , and it rules out  $k_{11} = \mathbb{F}_4$  in assertion (e) since  $|SL(2, \mathbb{F}_4)| = 60$ .

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